Introduction to Petri Nets

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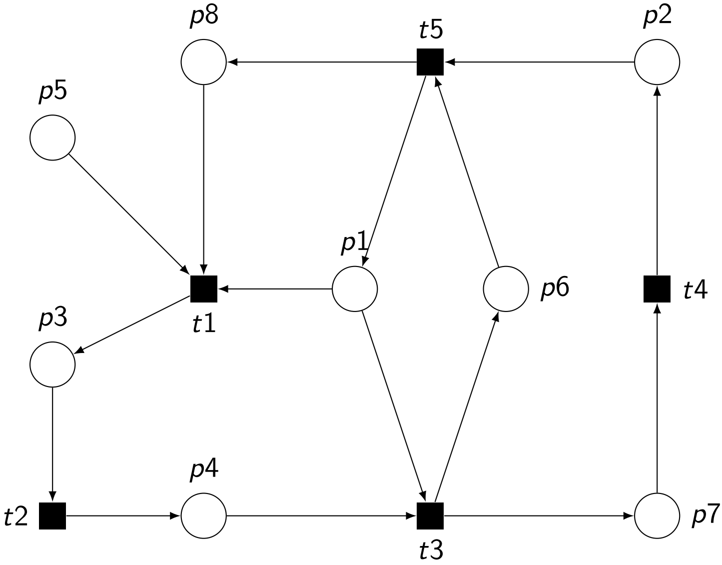
p8
t5
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Models of Concurrency

Petri Nets

Class of formal models of *concurrency*

- They are formal models:
  Directed graphs - “Distributed version of automata”.
- They model concurrency:
  Automata are *sequential*. Petri nets represent processes that can run “in parallel”.
- They are a whole class of models:
  Different types of Petri Nets for different problems.

Petri Net Systems

Petri Net + Dynamic Behaviour

- A Petri Net is a static structure “shape of a network”
- A Petri Net System can “run” - execute actions.

Motivation

Real Computers - μ-architecture

- CPU = Control Unit + Arithmetic Units + Registers
- Control Unit = Finite State Automaton
- Communication between units uses lines:
  Cable with a binary value
- Clock is crucial to know which message is on a line.
- $\Rightarrow$ Sequential Computation : Synchronous Systems

Distributed System (Several Computers)

- Different Clocks: Asynchronous Systems
- A CPU cannot always guess the state of other CPU's
- Relies on Communication Protocols
- The state of the system is determined by the local states of each CPU

Like puzzles? $\rightarrow$ www.nandgame.com

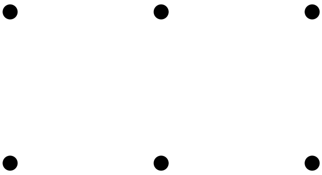
Basic Definitions

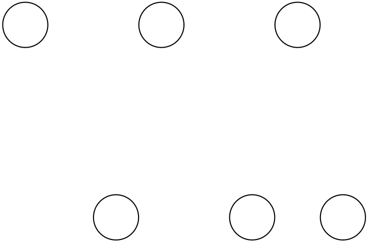
Automata

- States: Control
- Alphabet: Instructions
- Arcs (arrows): Effect of an instruction at a given state

Petri Nets

- Places: local states
- Transitions: change of state
- Arcs: Effect of a transition on the local states.





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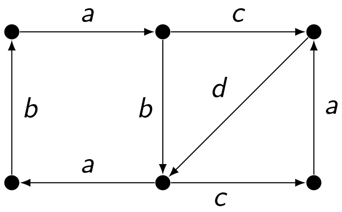

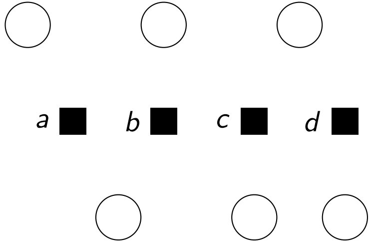

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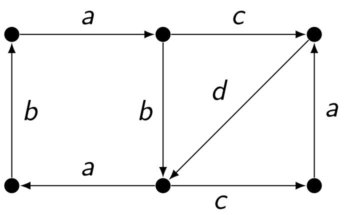

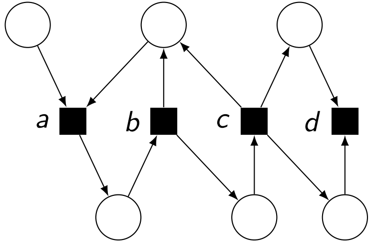

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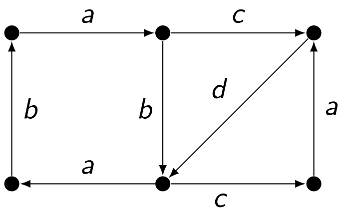

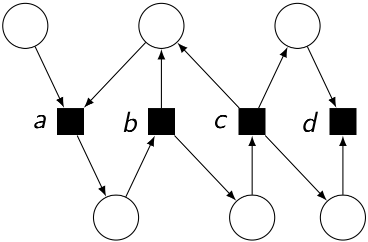

Q: What's missing?

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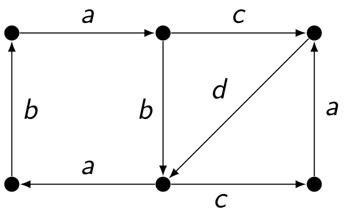
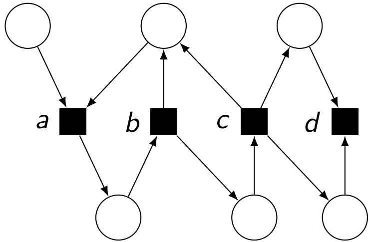




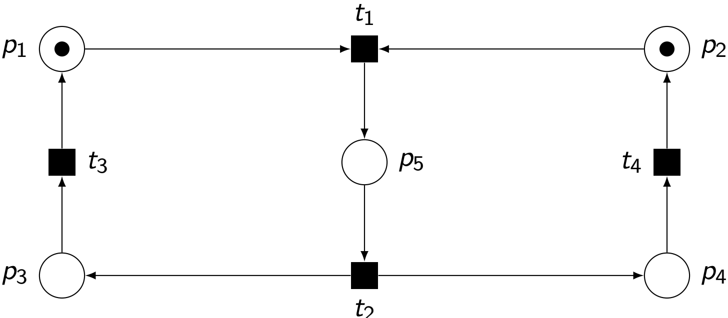
Q: What's missing? A: Initial state, accepting states

Most Basic Model

Elementary Net Systems

An E.N.S. is a tuple $\Sigma = (B, E, \mathcal{F}, m_0)$

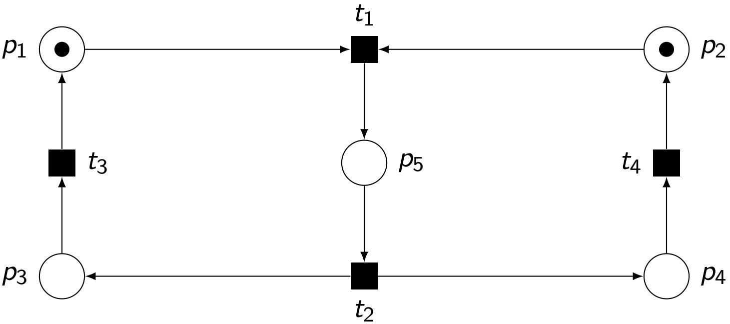
- B: Places $\rightarrow$ Conditions
- E: Transitions $\rightarrow$ Events
- $\mathcal{F} \subseteq (B \times E \cup E \times B)$: Arrows $\rightarrow$ Flow relation
- $m_0 : B \rightarrow \{0, 1\}$: Initial Marking (Global State): assigns 0 or 1 token to each condition.



Neighbourhoods of Events

Given an Event $e \in E$

- Pre-conditions: ${}^{\bullet}e = \{b \in B \mid (b, e) \in \mathcal{F}\}$
  conditions that "feed" the event
- Post-conditions: $e^{\bullet} = \{b \in B \mid (e, b) \in \mathcal{F}\}$
  conditions which are "fed" by the event.
- Two events e_1, e_2 are *independent* iff $({}^{\bullet}e_1 \cup e_1^{\bullet}) \cap ({}^{\bullet}e_2 \cup e_2^{\bullet}) = \emptyset$



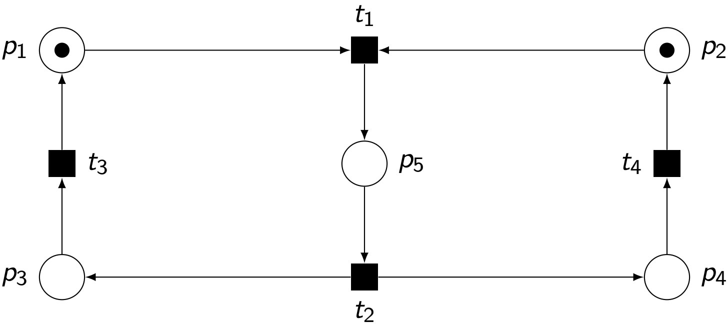
E.N.S. Dynamics

Enabled Events

Event e is enabled at marking m: $m[e\rangle$ iff:

- each $b \in {}^\bullet e$ has $m(b) = 1$ (all pre-conditions are true), AND
- each $b \in e^\bullet$ has $m(b) = 0$ (all post-conditions are false)

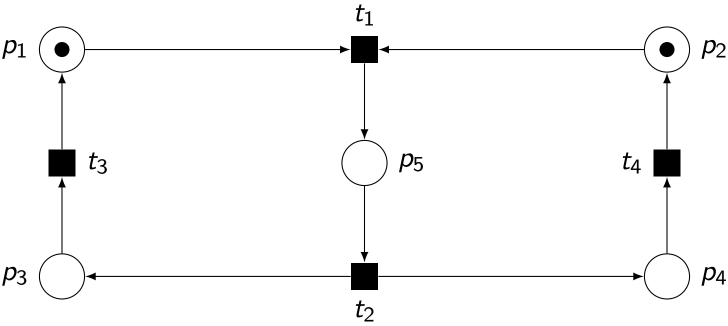
Two events e_1, e_2 are *independent* iff $({}^\bullet e_1 \cup e_1^\bullet) \cap ({}^\bullet e_2 \cup e_2^\bullet) = \emptyset$ Note: several events can be enabled at the same marking.



E.N.S. Dynamics

Firing Rule

- When an event is enabled, it may *fire*:
- $m_1[e\rangle m_2$ means that:
  - e is enabled at m_1, AND
  - $m_2 = (m_1 \setminus {}^{\bullet}e) \cup e^{\bullet}$



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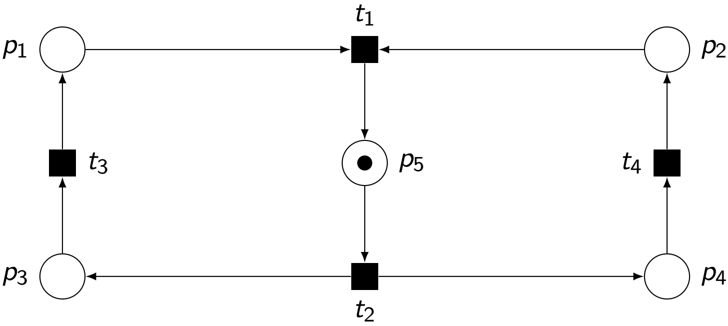

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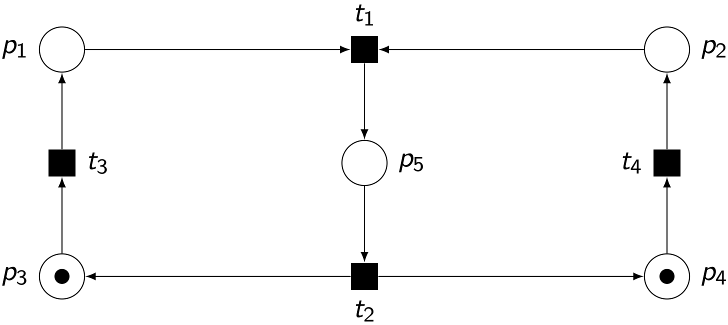
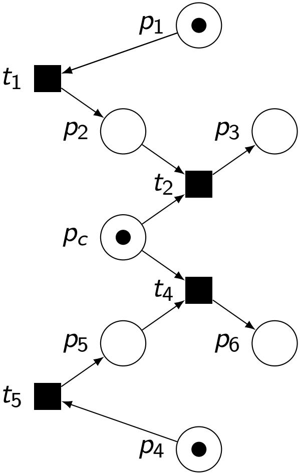
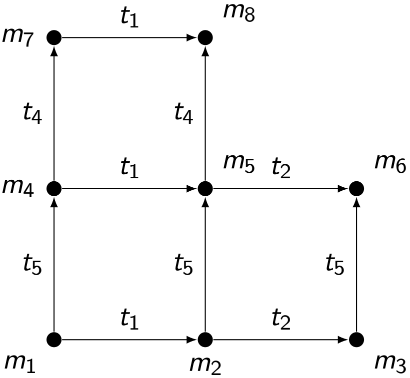


Global Behaviour: Automaton

Marking Graph

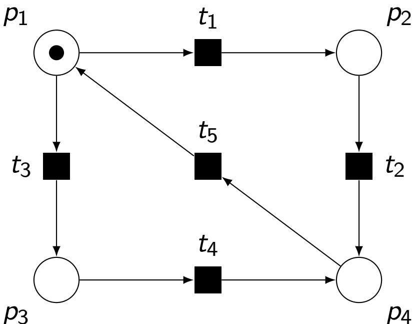
- Each node is a marking m
- We add an arc (m_1, m_2) with label e, if $m_1[e\rangle m_2$.

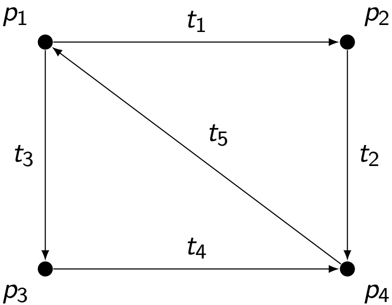




F.S.A.

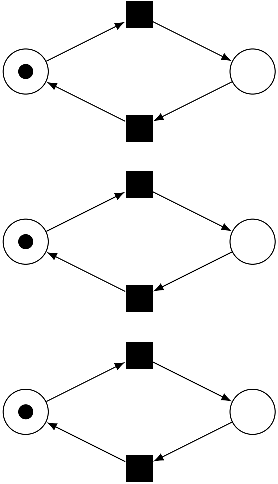
- E.N.S such that each event has ONLY 1 pre-condition and 1 post-condition
- only 1 token in the whole system
- then E.N.S. $\simeq$ Marking Graph
- Finite State Automaton



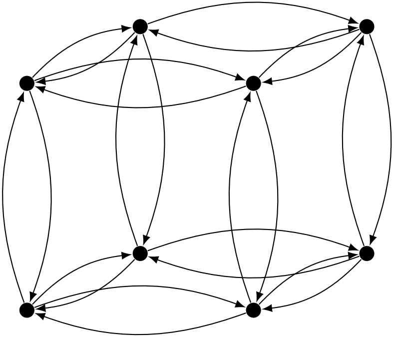


State Space Explosion

K components of size N



Size $= N * K$



Size$= N^K$

Motivation Part 2

Popularity

- Petri nets are widely used in the industry:
  System design: Software, Hardware, Logistics, etc. . .
- Two main reasons:
  - Modelling Power: Expressivity, Readability
  - Analysable: Algorithms for Verification

Modelling

Extensions of the basic model give flexibility.

Useful design tools.

• More features = More expressivity

Analysis

Good algorithms exist for verification.

Safety, Serviceability, Security, etc. . .

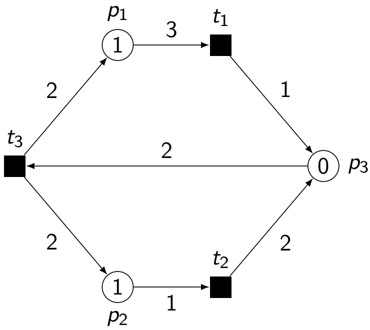
• Restrictions $\Rightarrow$ More algorithms (or more efficient).

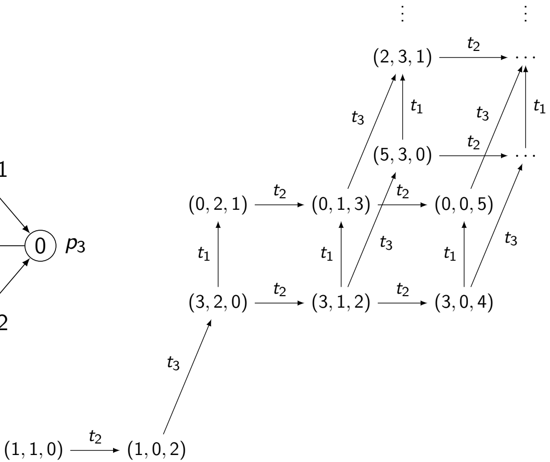
Place Transition Systems: Definitions

Place Transition Systems

- P: Places $\rightarrow$ Counters
- T: Transitions $\rightarrow$ Consume and Produce
- $\mathcal{F} : (B \times E \cup E \times B) \rightarrow \mathbb{N}$: Arcs are now weighted
- $m : P \rightarrow \mathbb{N}$: Marking assigns a number to each place
- Firing rule: $m_1[t\rangle m_2$
  - $\forall p \in P : \mathcal{F}(p, t) \leq m_1(p)$
    Places have enough tokens for the transition to fire, AND
  - $\forall p \in P : m_2(p) = m_1(p) - \mathcal{F}(p, t) + \mathcal{F}(t, p)$
    The weights on the arcs indicate how much is consumed and produced.

Place Transition Systems are VAS





$$t_1 = (-3, 0, 1); t_2 = (0, -1, 2); t_3 = (2, -2, 0)$$

Note: Unbounded Behaviour!

Problems

Boundedness

A P/T net system is *bounded* iff its set of reachable markings is finite.

- General case: PSPACE and PSPACE-complete if $|P| \geq 4$,
- PTIME for conflict-free nets (no choices)

Problems

Reachability

Given a P/T net system with initial marking m_0 and target marking m_t, decide whether m_t is reachable from m_0.

- In general, decidable but primitive recursive space!
- Undecidable if we allow for (at least) two zero-test arcs.
- 2EXPTIME if $|P| \leq 5$
- PSPACE-complete for 1-safe nets ($\simeq$ E.N.S)
- NP-complete for nets without cycles, and also for conflict-free nets (no choices).
- PTIME for bounded conflict-free nets
- PTIME for marked graphs
- PTIME for nets that are live, bounded, cyclic and free-choice.

Problems

Liveness

Deciding whether for any transition t, and from any reachable marking, there is another reachable marking that enables t.

- In general, primitive recursive equivalent to reachability, hence decidable.
- General complexity is an open problem.
- PSPACE-complete for 1-safe nets.
- co-NP-complete free-choice nets.
- PTIME for bounded bounded free-choice nets
- PTIME for conflict-free nets

Deadlock-freedom

A net is *deadlock-free* iff every reachable marking enables some transition.

- In general, reduction to reachability in PTIME
- PSPACE-complete for 1-safe nets
- NP-complete 1-safe free-choice nets
- PTIME for conflict-free nets.